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ON LAMINAR PRESEPARATION FLOW*

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The boundary layer of an incompressible fluid in the domain ahead of the departure of the free streamline from the surface of a smooth body or a break-point of its generator, is considered. The potential of the external irrotational velocity field is taken from the theory of jet flows. It is assumed with respect to the initial value of the surface friction that its order can vary over a wide range, while remaining finite, or taking extremely large values. The boundary layer in the preseparation domain always admits of a unified mathematical treatment, in which the initial surface friction plays the role of a parameter.

1. External potential flow.

For measuring both the independent and the required quantities we take a system of units in which the basis quantities are the radius of curvature of the body generator at the point of separation, the velocity of the external potential flow at this point, and the fluid density. Changing to dimensionless variables, we direct the s axis of the curvilinear orthogonal system of coordinates along the body generator, and the n axis along the normal to it. Let u', v' be the components of the disturbing velocity vector, and p' the excess pressure in the external potential flow domain. In accordance with the linearized form of the Bernoulli integral, $u'=-p'$, while the complex velocity is $-(p'+iv')$. By the theory of jet flows of an ideal incompressible fluid, we know that, in the neighbourhood of the departure point of the free streamline from the body /1/

$$p'+iv'=ib_{1/2}z^{1/2}+ib_{3/2}z^{3/2}+\ldots,\quad z=s+in \tag{1.1}$$

When $\arg z\to 0$, the pressure $p'\to 0$, whereas

$$v'=b_{1/2}s^{1/2}+b_{3/2}s^{3/2}+\ldots \tag{1.2}$$

If $\arg z\to\pi$, then $v'\to 0$, while

$$p'\to -b_{1/2}(-s)^{1/2}+b_{3/2}(-s)^{3/2}+\ldots \tag{1.3}$$

In accordance with (1.2), the equation of the free streamline is

$$n=\frac{2}{3}b_{1/2}s^{3/2}+\frac{2}{5}b_{3/2}s^{5/2}+\ldots \tag{1.4}$$

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It was shown by Sychev /2/ that, for the boundary layer when the flow past the body is poor, the constant $b_{1/2}$ is positive and of the order of $R^{-1/16}$, where R is the Reynolds number. This estimate implies justification of the so-called Brillouin-Ville condition, according to which, in the limit as $R \to \infty$, the curvature of the free streamline at the point of departure from the body is equal to the curvature of the body contour. For, see (1.4), the curvature of the fluid jet enveloping the body then following the boundary of the stagnant zone, remains continuous at the point $s = 0$ as $b_{1/2} \to 0$.

The possibility of applying the results of jet theory to a local description of the velocity field on separation was also studied by Messiter and Enlow /3/, and led them to the incorrect claim that $b_{1/2} = 0$ for any large (but nowhere infinite) Reynolds number. In their expansion of the complex velocity, the series in half-integral powers of z started with a term proportional to $z^{3/2}$. Later calculation of this term /4, 5/ did not change Sychev's conclusions.

The constants $b_{1/2}$ and $b_{3/2}$ of (1.1) are determined by the global flow picture, which may in fact be cavitational. In the latter case their values are changed on changing from one model to another, though the changes are slight /6/. In the context of local theory, aimed at constructing the velocity field close to the separation of the flow from the body surface, both constants have to be specified in such a way that account is taken of the typical situations that arise in the flow past obstacles of different shape, in which the conditions of boundary layer formation in the preseparation zone may prove to be very different from case to case.

2. Transformation of the Prandtl equations. Denoting the transverse coordinate and stream function, normalized in the usual way, for the boundary layer, by N and Ψ respectively, we write

$$\frac{\partial\Psi}{\partial N}\frac{\partial^2\Psi}{\partial s\partial N} - \frac{\partial\Psi}{\partial s}\frac{\partial^2\Psi}{\partial N^2} = \frac{\partial^3\Psi}{\partial N^3} - \frac{dp'}{ds} \tag{2.1}$$

where the excess pressure p' is given by (1.3). We shall assume that the constant $b_{1/2}$ in it can have either sign, while the constant $b_{3/2} \sim 1$ is positive. A further parameter is extremely important for describing the boundary layer, namely, the characteristic size of the surface friction λ. Henceforth, the constants $b_{1/2}$ and λ are assumed to be subject to at least one of the two inequalities $|b_{1/2}| \ll 1$, or $\lambda \gg 1$.

When $|b_{1/2}| \ll 1$ and $\lambda \sim 1$, the Prandtl equation in canonical form (2.1) is initial. But when $|b_{1/2}| \sim 1$ and $\lambda \gg 1$, we first have to change the independent variables and the required stream function in it, since the effective thickness of the boundary layer is greatly reduced. The key to this change is the invariance of (2.1) (with the derivative dp'/ds neglected) under the two-parameter group of similarity transformations

$$s = \alpha x, \quad N = \beta y_2, \quad \Psi = \gamma\psi_2, \quad \gamma = \alpha\beta^{-1} \tag{2.2}$$

Putting $\alpha = \lambda^{-2}$, $\beta = \lambda^{-1}$, with the resulting equation $\gamma = \lambda^{-1}$, we find that, in the new variables, the characteristic size of the surface friction is unity, while the excess pressure is

$$p' = \lambda^{-1}[-b_{1/2}(-x)^{1/2} + \lambda^{-2}b_{3/2}(-x)^{3/2} + \ldots] \tag{2.3}$$

We shall assume that the changes (2.2) have been made, i.e., we have changed to a system of measuring units in which all the boundary layer parameters retain finite values, while the excess pressure is small.

3. Viscous flow in the wall layer. Due to the singularity in the pressure gradient given by (2.3), the boundary layer has to be divided into two domains: its main thickness, and the thin layer immediately adjacent to the wall /2, 3/. In the first domain the flow can be regarded as locally inviscid, while in the second, if we ignore viscous tangential stresses, we cannot satisfy the condition of fluid adhesion on the solid surface. To isolate the main term in the expression for the pressure, with any values of $b_{1/2}$ and λ, it is natural to make the extension $x = |b_{1/2}|\lambda^{-1}x_1$ of the longitudinal coordinate, after which we introduce the transverse coordinate y_3 and stream function ψ_3 for the velocity field in the viscous sublayer by

$$y_2 = |b_{1/2}|^{1/3}\lambda^{-1/3}y_3, \quad \psi_2 = |b_{1/2}|^{2/3}\lambda^{-2/3}\psi_3 \tag{3.1}$$

In these variables, (2.1) becomes

$$\frac{\partial\psi_3}{\partial y_3}\frac{\partial^2\psi_3}{\partial x_1\partial y_3} - \frac{\partial\psi_3}{\partial x_1}\frac{\partial^2\psi_3}{\partial y_3^2} = \frac{\partial^3\psi_3}{\partial y_3^3} - |b_{1/2}|^{5/6}\lambda^{-5/6}\frac{dp_1}{dx_1} \tag{3.2}$$

Since the normalized excess pressure p_1 is given by

$$p' = |b_{1/2}|^{3/2}\lambda^{-3/2}p_1, \quad p_1 = -\operatorname{sign} b_{1/2}(-x_1)^{1/2} + b_{3/2}\lambda^{-3}(-x_1)^{3/2} + \ldots \tag{3.3}$$

it is obvious that the term with its gradient on the right-hand side of (3.2) is small, not only with $|b_{1/2}| \ll 1$ and $\lambda \sim 1$, but also with $|b_{1/2}| \sim 1$ and $\lambda \gg 1$. This property, which is justified afresh by the invariance of the Prandtl equation (without the derivative dp'/dx) under the group of similarity transformations, enables the solution of (3.2) to be expressed by means of the asymptotic sequence

$$\psi_3 = {}^1/_2 y_3^2 + |b_{1/_2}|^{3/_4} \lambda^{-5/_4} [\psi_3^{(1/_2)}(x_1, y_3) + \lambda^{-3}\psi_3^{(3/_2)}(x_1, y_3)] + b_{5/_3}\psi_3^{(5/_3)}(x_1, y_3) + \ldots \tag{3.4}$$

Here, the constant $b_{5/_3}$ can be chosen arbitrarily, as will be seen below; the term with the function $\psi_3^{(3/_2)}$ has to preserved, if $\lambda \sim 1$. Without specially isolating the last case, each of the terms $\psi_3^{(m)}$, $m = {}^1/_2, {}^3/_2, {}^5/_3$ in (3.4) will be represented by the equation

$$\psi_3^{(m)} = (-x_1)^m f^{(m)}(\xi), \quad \xi = (-x_1)^{-1/_3} y_3 \tag{3.5}$$

The ordinary differential equation for $f^{(m)}$ is

$$\frac{d^3 f^{(m)}}{d\xi^3} - \frac{1}{3}\xi^2 \frac{d^2 f^{(m)}}{d\xi^2} + m\xi\frac{df^{(m)}}{d\xi} - mf^{(m)} = m\delta_m \tag{3.6}$$

where $\delta_m = \operatorname{sign} b_{1/_2}$ for $m = {}^1/_2$, $\delta_m = -b_{3/_2}$ for $m = {}^3/_2$ and $\delta_m = 0$ for $m = {}^5/_3$. The boundary conditions

$$f^{(m)} = df^{(m)}/d\xi = 0 \quad \text{for} \quad \xi = 0 \tag{3.7}$$

for Eq.(3.6) are obtained from the fluid/body adhesion condition. Moreover, there must be no terms in the solution which increases exponentially as $\xi \to \infty$.

Differentiating (3.6) and substituting $g^{(m)} = d^2 f^{(m)}/d\xi^2$, we find, following /3/, that the new required quantity, regarded as the function $\eta = \xi^3/9$, satisfies the homogeneous equation

$$\eta\frac{d^2 g^{(m)}}{d\eta^2} + \left(\frac{2}{3} - \eta\right)\frac{dg^{(m)}}{d\eta} + \left(m - \frac{2}{3}\right)g^{(m)} = 0 \tag{3.8}$$

Subsituting (3.7) into (3.6), we obtain the boundary condition

$$dg^{(m)}/d\eta = 3^{-1/_3} m\delta_m \eta^{-2/_3} \quad \text{as} \quad \eta \to 0 \tag{3.9}$$

for Eq.(3.8). To eliminate the exponential increase in the solution at infinity, we put

$$g^{(m)} = A_m \Psi\,({}^2/_3 - m, {}^2/_3; \eta) \tag{3.10}$$

with the Tricomi Ψ-function /7/ on the right-hand side. As $\eta \to \infty$, we have

$$g^{(m)} = A_m [\eta^{m-2/_3} - (1-m)({}^2/_3 - m)\eta^{m-5/_3} + \ldots] \tag{3.11}$$

Denoting by Γ and Φ respectively the Euler gamma function and the confluent hypergeometric function, we rewrite (3.10) as /7/

$$g^{(m)} = A_m\left[\frac{\Gamma({}^1/_3)}{\Gamma(1-m)}\Phi\left(\frac{2}{3} - m, \frac{2}{3}; \eta\right) + \frac{\Gamma(-{}^1/_3)}{\Gamma({}^2/_3 - m)}\eta^{1/_3}\Phi\left(1 - m, \frac{4}{3}; \eta\right)\right] \tag{3.12}$$

Since $\Phi = 1$ for $\eta = 0$, boundary condition (3.9) gives

$$\frac{\Gamma(-{}^1/_3)}{\Gamma({}^2/_3 - m)} A_m = 3^{1/_3} m\delta_m \tag{3.13}$$

For $m = {}^1/_2, {}^3/_2$, (3.13) gives the constants

$$A_{1/_2} = -\frac{1}{2\cdot 3^{2/_3}}\frac{\Gamma({}^1/_6)}{\Gamma({}^2/_3)}\operatorname{sign} b_{1/_2}, \quad A_{3/_2} = -\frac{3^{4/_3}}{5}\frac{\Gamma({}^1/_6)}{\Gamma({}^2/_3)} b_{3/_2} \tag{3.14}$$

on recalling that $\delta_m = \operatorname{sign} b_{1/_2}$ for $m = {}^1/_2$ and $\delta_m = -b_{3/_2}$ for $m = {}^3/_2$. However, with $\delta_m = 0$ we can only obtain a non-trivial value of the constant A_m in the case when

$$m = {}^2/_3 + N, \quad N = 0, 1, 2, \ldots \tag{3.15}$$

The first eigenvalue of spectrum (3.15) is $m = {}^2/_3$, which generates the eigenfunction $g^{(2/_3)} = 1$, corresponding to a shift in the initial surface friction λ, and hence is of no interest. The second eigenvalue $m = {}^5/_3$ justifies the introduction into expansion (3.4) of the term proportional to the constant $b_{5/_3}$. Since this constant is not contained in the external expansion (1.1)-(1.4) for the potential flow, we put $A_{5/_3} = -{}^3/_2$. Then, the second eigenfunction is $g^{(5/_3)} = 1 - {}^3/_2\,\eta$, and corresponding to it we have

$$f^{(5/_3)} = {}^1/_2(\xi^2 - {}^1/_{60}\xi^5)$$

which gives a typical velocity field for a boundary layer with zero pressure gradient.

Substituting the values (3.14) of the constants $A_{1/_2}$ and $A_{3/_2}$ into (3.12), we can find the surface friction τ_w acting on the body. In our dimensionless normalized variables:

$$\tau_w = \frac{\partial^2\psi_2}{\partial y_2^2}\bigg|_{y_2=0} = \frac{\partial^2\psi_3}{\partial y_3^2}\bigg|_{y_3=0} = 1 + \frac{\Gamma({}^1/_6)\Gamma({}^1/_3)}{2\cdot 3^{1/_3}\pi^{1/_2}\Gamma({}^2/_3)} \times \tag{3.16}$$
$$|b_{1/_2}|^{3/_4}\lambda^{-5/_4}\left[-\operatorname{sign} b_{1/_2}(-x_1)^{-1/_6} + \frac{9}{5} b_{3/_2}\lambda^{-3}(-x_1)^{5/_6}\right] + b_{5/_3}(-x_1) + \ldots$$

It remains to find the behaviour of the fluid parameters on the outer edge $y_3 \to \infty$ of the viscous sublayer, whence we can find the conditions for union with the stream function ψ_2.

Returning to (3.5) and (3.11), we have

$$\begin{aligned}\psi_3 = {} & \frac{1}{2} y_3^2 + |b_{1/2}|^{5/6} \lambda^{-5/6} \Big\{\operatorname{sign} b_{1/2} \times \\ & \Big[-\frac{2\Gamma(1/6)}{3\Gamma(2/3)} y_3^{3/2} + B_{1/2}(-x_1)^{1/6} y_3 - (-x_1)^{1/2} + \ldots\Big] - \\ & b_{5/2}\lambda^{-3}\Big[\frac{4\Gamma(1/6)}{315\Gamma(2/3)} y_3^{9/2} - \frac{\Gamma(1/6)}{\Gamma(2/3)}(-x_1) y_3^{3/2} + B_{3/2}(-x_1)^{7/6} y_3 - \\ & (-x_1)^{3/2} + \ldots\Big]\Big\} + \frac{1}{2} b_{5/2}\Big[-\frac{1}{60} y_3^5 + (-x_1) y_3^2\Big] + \ldots\end{aligned} \tag{3.17}$$

where $B_{1/2}$ and $B_{3/2}$ are constants which arise on integrating $g^{(m)}$ with $m = 1/2, 3/2$.

4. The main thickness of the boundary layer. Transforming in the Prandtl equation to the variables x_1, y_2, ψ_2 and recalling (3.3) for the excess pressure, we write it as

$$\frac{\partial\psi_2}{\partial y_2}\frac{\partial^2\psi_2}{\partial x_1 \partial y_2} - \frac{\partial\psi_2}{\partial x_1}\frac{\partial^2\psi_2}{\partial y_2^2} = |b_{1/2}|\lambda^{-1}\frac{\partial^3\psi_2}{\partial y_2^3} - |b_{1/2}|^{3/2}\lambda^{-3/2}\frac{dp_1}{dx_1} \tag{4.1}$$

As distinct from (3.2), both the coefficients of the terms on the right-hand side are expressible in terms of the ratio $|b_{1/2}|/\lambda$, which tends to zero, both when $|b_{1/2}| \ll 1$ and $\lambda \sim 1$, and when $|b_{1/2}| \sim 1$ and $\lambda \gg 1$. Under this condition, the principal feature in (4.1) becomes the operator on the left-hand side, generated by taking account of the inertia forces in the Prandtl equation, while the contributions from the pressure gradient and the viscous tangential stresses are small. In order to be able to satisfy the condition for union of ψ_2 with the stream function for the narrow wall sublayer, with $|b_{1/2}|/\lambda \ll 1$ it is natural to seek the solution as the asymptotic sequence

$$\begin{aligned}\psi_2 = {} & \psi_{21}^{(0)}(y_2) + |b_{1/2}|\lambda^{-1}\psi_{22}(x_1, y_2) + |b_{1/2}|^{7/6} \times \\ & \lambda^{-7/6}\psi_{23}(x_1, y_2) + |b_{1/2}|^{3/2}\lambda^{-3/2}\psi_{24}(x_1, y_2) + \ldots\end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1), we find

$$\frac{d\psi_{21}^{(0)}}{dy_2}\frac{\partial^2\psi_{2l}}{\partial x_1 \partial y_2} - \frac{d^2\psi_{21}^{(0)}}{dy_2^2}\frac{\partial\psi_{2l}}{\partial x_1} = \Psi_l \tag{4.3}$$

where the function $\psi_{21}^{(0)}$ is left arbitrary. The index $l = 2$, for which $\Psi_l = d^3\psi_{21}^{(0)}/dy_2^3$, distinguishes the regular term in the solution of the Prandtl equation. It is obvious in advance that, with $l = 3$, the right-hand side $\Psi_l = 0$, so that ψ_{23} is in essence the eigenfunction defined by the inertia forces of the operator on the left-hand side of (4.1). Finally, corresponding to the singularities in the pressure gradient (3.3) we have the index $l = 4$ with $\Psi_l = -dp_1/dx_1$.

We seek the solution of Eqs.(4.3) as the expansions

$$\begin{aligned}\psi_{22} &= \psi_{22}^{(0)}(y_2) + (-x_1)\psi_{22}^{(1)}(y_2) + \ldots \\ \psi_{23} &= (-x_1)^{1/6}[\psi_{23}^{(1/6)}(y_2) + (-x_1)\psi_{23}^{(7/6)}(y_2) + \ldots] \\ \psi_{24} &= (-x_1)^{1/2}[\psi_{24}^{(1/2)}(y_2) + (-x_1)\psi_{24}^{3/2}(y_2) + \ldots]\end{aligned} \tag{4.4}$$

with respect to the longitudinal coordinate. Like $\psi_{21}^{(0)}$, we choose $\psi_{22}^{(0)}$ arbitrarily, while the remaining functions on the right-hand sides of (4.4) satisfy the ordinary differential equations

$$\frac{d\psi_{21}^{(0)}}{dy_2}\frac{d\psi_{2l}^{(m)}}{dy_2} - \frac{d^2\psi_{21}^{(0)}}{dy_2^2}\psi_{2l}^{(m)} = \Psi_l^{(m)} \tag{4.5}$$

where the right-hand sides $\Psi_l^{(m)}$ are evaluated from Ψ_l of (4.3).

It is clear from the first of Eqs.(3.1) that the variable $y_2 \to 0$ if we fix $y_3 \sim 1$. Conversely, let $y_2 \sim 1$ be assumed given; then $y_3 \to \infty$. These relations show how the solutions $\psi_2(x_1, y_2)$ and $\psi_3(x_1, y_3)$ must be united for the two domains into which the boundary layer splits up. Using the second of Eqs.(3.1), relation (3.17) first establishes the limits

$$\begin{aligned}\psi_{21}^{(0)} &= \frac{1}{2} y_2^2 - \frac{4\Gamma(1/6)}{315\Gamma(2/3)} b_{5/2}\lambda^{-3} y_2^{9/2} - \frac{1}{120}|b_{1/2}|^{-1}\lambda b_{5/2} y_2^5 + \ldots \\ \psi_{22}^{(0)} &= -\frac{2\Gamma(1/6)}{3\Gamma(2/3)} \operatorname{sign} b_{1/2} y_2^{3/2} + \ldots\end{aligned} \tag{4.6}$$

to which the functions $\psi_{21}^{(0)}$ and $\psi_{22}^{(0)}$, which have been left arbitrary, tend as $y_2 \to 0$. The first of Eqs.(4.6) shows that the constant $b_{5/2} = |b_{1/2}|\lambda^{-1}B_{5/2}$ with $B_{5/2} \sim 1$. When $\lambda \gg 1$, the second term on its right-hand side becomes small compared with the others, and can be omitted.

The regular term in the solution of the Prandtl equation is fixed by the limit

$$\psi_{22}^{(1)} = \frac{\Gamma(1/3)}{\Gamma(2/3)} b_{3/2} \lambda^{-3} y_2^{4/3} + \frac{1}{2} B_{3/2} y_2^2 + \ldots \tag{4.7}$$

The eigenfunctions of the operator, defined by the inertia forces, on the left-hand side of (4.1) have the limits

$$\psi_{23}^{(1/4)} = B_{1/2} \operatorname{sign} b_{1/2} y_2 + \ldots, \quad \psi_{23}^{(7/4)} = -B_{3/2} b_{3/2} \lambda^{-3} y_2 + \ldots \tag{4.8}$$

Finally, the limiting expressions of the functions due to the singularities in the pressure gradient are

$$\psi_{24}^{(1/2)} = (-1) \operatorname{sign} b_{1/2} + \ldots, \quad \psi_{24}^{(3/2)} = b_{3/2} \lambda^{-3} + \ldots \tag{4.9}$$

In the case $\lambda \gg 1$, the first term on the right of (4.7) can be neglected, and in addition, the limits of $\psi_{23}^{(7/4)}$, $\psi_{24}^{(3/2)}$ as $y_2 \to 0$ can be assumed to be zero. This is equivalent to neglecting, right from the start, the term with $(-x_1)^{3/2}$ in relation (3.3) for the excess pressure. If $b_{3/2} \ll 1$ and $\lambda \sim 1$, then all the terms in the limit relations (4.6)–(4.9) are of the same order, though this does not change our conclusions for the flow domain considered, which are based on Sychev's analysis /2/.

The solution of Eq.(4.5) which satisfies condition (4.7) is

$$\psi_{22}^{(1)} = -\frac{d\psi_{21}^{(0)}}{dy_2} \int_0^{y_2} \left(\frac{d\psi_{21}^{(0)}}{dy} \right)^{-2} \frac{d^3\psi_{21}^{(0)}}{dy^3} dy \tag{4.10}$$

The integrals of Eq.(4.5), which turn into linear functions (4.8) as $y_2 \to 0$, are

$$\psi_{23}^{(1/4)} = B_{1/2} \operatorname{sign} b_{1/2} \frac{d\psi_{21}^{(0)}}{dy_2}, \quad \psi_{23}^{(7/4)} = -B_{3/2} b_{3/2} \lambda^{-3} \frac{d\psi_{21}^{(0)}}{dy_2} \tag{4.11}$$

Solving Eq.(4.5) with limiting values (4.9), we obtain

$$\psi_{24}^{(1/2)} = \operatorname{sign} b_{1/2} \left\{ \frac{d\psi_{21}^{(0)}}{dy_2} \int_0^{y_2} \left[\left(\frac{d\psi_{21}^{(0)}}{dy} \right)^{-2} - \frac{1}{y^2} \right] dy - \frac{1}{y_2} \frac{d\psi_{21}^{(0)}}{dy_2} \right\} \tag{4.12}$$

$$\psi_{24}^{(3/2)} = -b_{3/2} \lambda^{-3} \left\{ \frac{d\psi_{21}^{(0)}}{dy_2} \int_0^{y_2} \left[\left(\frac{d\psi_{21}^{(0)}}{dy} \right)^{-2} - \frac{1}{y^2} \right] dy - \frac{1}{y_2} \frac{d\psi_{21}^{(0)}}{dy_2} \right\}$$

Eqs.(4.10)–(4.12) complete the construction of the velocity field in the basic thickness of the boundary layer. It must be said that the quantities $\psi_{23}^{(1/4)}$, $\psi_{23}^{(7/4)}$, introduced by Eqs. (4.11), though due to the presence of singularities in the external pressure distribution, are formally eigenfunctions of the operator of inertia forces on the left-hand side of (4.1).

5. The non-linear domain. We return to Eq.(3.16) for the surface friction, which shows that, when $(-x_1) \sim |b_{1/2}|^5 \lambda^{-5}$ the first correction term on the right-hand side becomes of order unity. This characteristic size is inherent in the non-linear domain, where our above analysis loses its force. We mark all the quantities in this domain with a bar, and in accordance with our remark, we normalize the longitudinal coordinate as follows:

$$x_1 = |b_{1/2}|^5 \lambda^{-5} \bar{x} \tag{5.1}$$

In the viscous wall layer located here, as follows from the second relation (3.5) for the similarity variable ξ, the transverse coordinate $\bar{y} \sim (-\bar{x})^{1/3}$. As a result, we have

$$y_3 = |b_{1/2}|^{5/3} \lambda^{-5/3} \bar{y} \tag{5.2}$$

Since $\bar{\psi} \sim \bar{y}^2$, then

$$\psi_3 = |b_{1/2}|^{10/3} \lambda^{-10/3} \bar{\psi} \tag{5.3}$$

We can arrive at the same estimate by noting that the term with $\psi_3^{(1/2)}$ on the right-hand side of (3.5), which has played the role of a small disturbance, becomes of the order of the basic solution (the terms $\psi_3^{(3/2)}$, $\psi_3^{(5/2)}$ give a start to higher appoximations). This conclusion is confirmed by comparing the terms with $\psi_{21}^{(0)}$, $\psi_{22}^{(0)}$ in expansion (4.2) for the main thickness of the boundary layer.

Substituting (5.1) into the second of (3.3), we obtain the definition

$$p_1 = |b_{1/2}|^{5/2} \lambda^{-5/2} \bar{p} \tag{5.4}$$

of the excess pressure. Now, all the variables, both independent and required, are expressible in terms of the parameters $|b_{1/2}|$ and λ.

On referring it to the new variables, the Prandtl Eq.(3.2) becomes

$$\frac{\partial \bar{\psi}}{\partial \bar{y}} \frac{\partial^2 \bar{\psi}}{\partial \bar{x} \partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{x}} \frac{\partial^2 \bar{\psi}}{\partial \bar{y}^2} = \frac{\partial^3 \bar{\psi}}{\partial \bar{y}^3} - \frac{d\bar{p}}{d\bar{x}} \tag{5.5}$$

where there are no terms with small coefficients. The initial conditions are obtained for it by union of the stream function $\bar\psi(\bar x, \bar y)$ with $\psi_3(x_1, y_3)$ and of the pressure $\bar p(\bar x)$ with $p_1(x_1)$. Since the similarity variable $\xi = (-\bar x)^{-1/3}\bar y$ remains invariant under affine transformation (5.1), (5.2), the conditions

$$\bar\psi \to {}^1/_2\bar y^2 + (-\bar x)^{1/2} f_3^{(1/2)}(\xi) + \ldots, \quad p \to -\operatorname{sign} b_{1/2}(-\bar x)^{1/2} + \ldots \tag{5.6}$$

which appear as $\bar x \to -\infty$, $\xi = \text{const}$, are the same as those obtained at the input of the boundary layer, freely interacting with the external potential flow /2, 8, 9/.

To state the boundary conditions for Eq.(5.5) as $\bar y \to \infty$, we have to construct the solution for the main thickness of the boundary layer in the non-linear domain. In $\bar x, y_2, \psi_2$, variables Eq.(4.1) becomes

$$\frac{\partial\psi_2}{\partial y_2}\frac{\partial^2\psi_2}{\partial\bar x\,\partial y_2} - \frac{\partial\psi_2}{\partial\bar x}\frac{\partial^2\psi_2}{\partial y_2^2} = |b_{1/2}|^6\lambda^{-6}\frac{\partial^3\psi_2}{\partial y_2^3} - |b_{1/2}|^4\lambda^{-4}\frac{d\bar p}{d\bar x} \tag{5.7}$$

and therefore, in order to integrate it, we can use an asymptotic expansion in powers of $|b_{1/2}|/\lambda$. The union with the stream function (4.2) shows that there must be linear and quadratic terms in the required expansion. Repeating the arguments of Sect.4 in the light of this remark, we obtain

$$\psi_2 = \psi_{21}^{(0)}(y_2) + |b_{1/2}|\lambda^{-1}\psi_{22}^{(0)}(y_2) + |b_{1/2}|^2\lambda^{-2}A(\bar x)\frac{d\psi_{21}^{(0)}}{dy_2} + \ldots \tag{5.8}$$

The term with displacement thickness A represents the eigenfunction of the operator on the left-hand side of (5.7), determined by the inertia forces. The choice of A remains arbitrary, except that $A \to B_{1/2}\operatorname{sign} b_{1/2}\times(-\bar x)^{1/6}$, $\bar x \to -\infty$.

The union of the stream function $\bar\psi(\bar x, \bar y)$, normalized by (5.3), with the solution (5.8) for $\psi_2(\bar x, y_2)$, leads to the condition

$$\bar\psi \to \frac{1}{2}\bar y^2 - \frac{2\Gamma(1/6)}{3\Gamma(2/3)}\operatorname{sign} b_{1/2}\,\bar y^{3/2} + A(\bar x)\bar y + \ldots \quad \text{as } \bar y \to \infty \tag{5.9}$$

which repeats the condition arising at the edge of the viscous wall layer in the domain of free interaction /2, 8, 9/.

It remains to consider the external potential fluid motion, where the transverse coordinate y_e has a size of the same order as the longitudinal coordinate. Since $s = |b_{1/2}|^6\lambda^{-8}\bar x$ by (5.1), then also $n = |b_{1/2}|^6\lambda^{-8}y_e$. Instead of the stream function, it is convenient here to operate directly with components $u' = |b_{1/2}|^4\lambda^{-4}u_e$ and $v' = |b_{1/2}|^4\lambda^{-4}v_e$ of the disturbed velocity vector. A complete analysis of the potential motion is beyond the scope of the present paper; to see the underlying ideas, we need only quote the condition

$$v_e = -KdA/d\bar x \text{ for } y_e = 0, \quad K = R^{-1/2}|b_{1/2}|^{-8}\lambda^9 \tag{5.10}$$

which follows from the union with the velocity field in the main thickness of the boundary layer.

The constant K is a similarity parameter, defining the flow mode. With $K \ll 1$, we obtain for the complex velocity the problem of flow past the initial body, since in accordance with (5.10) we have $v_e(\bar x, 0) = 0$ to a first approximation. The excess pressure is then found from its limiting value at infinity. Hence, $\bar p = -\operatorname{sign} b_{1/2}(-\bar x)^{1/2} + \ldots$, whereas the displacement thickness $A(\bar x)$ is boundary condition (5.9) remains unknown. It is found during integration of Eq.(5.5), which starts as $\bar x \to -\infty$ with the initial distribution for the stream function, assigned by the first of relations (5.6).

Now let $K \sim 1$. This is the mode of free interaction of the boundary layer /2, 8, 9/. Since the function A in (5.10) is not known in advance, the selfinduced pressure $\bar p$ introduced by (5.4) is one of the required functions with singular behaviour as $\bar x \to -\infty$. The relation $K \sim 1$ fixes the connection between basic parameters $|b_{1/2}|$ and λ, specifying the pressure gradient and characteristic surface friction in the boundary layer respectively, and the Reynolds number R. When $\lambda \sim 1$, the coefficient $|b_{1/2}| \sim R^{-1/16} \ll 1$, which repeats the result of /2/ for flow past a smooth body. If $|b_{1/2}| \sim 1$, then $\lambda \sim R^{1/18} \gg 1$ with the consequent estimates of sizes $s = |b_{1/2}|^6\lambda^{-8}(-\bar x) \sim R^{-4/9}$, $n = R^{-1/2}|b_{1/2}|^2 \times \lambda^{-3}\bar y \sim R^{-2/3}$ of the wall layer in the domain of free interaction, which were obtained in /10/ in accordance with the previously proposed description of the field of preseparation flow /11/. Simple analysis of the solution of /11/ shows that the surface friction τ_w in it is subject to the relation $K \sim 1$, if λ is replaced by τ_w.

Notice finally that the change of sign of the coefficient of $b_{1/2}$ from positive to negative when $|b_{1/2}| \sim R^{-1/16}$ and $\lambda \sim 1$ represents a passage from flows with separation from a body of poor shape, considered in /2/, to jet flows, in which the free streamlines issue from the ends of small arcs or corner points of the body with breaks of the generator. When $b_{1/2} > 0$, the Brillouin-Ville points are located on the surface of the small arcs with end pieces immersed in the stagnant zone. Shortening of the arcs leads to a drop in $b_{1/2}$, but the case $b_{1/2} \ll R^{-1/16}$ requires special analysis. If $b_{1/2} < 0$, there are no Brillouin-Ville points on the arcs.

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INVESTIGATION OF SELFSIMILAR SOLUTIONS DESCRIBING FLOWS IN MIXING LAYERS*

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A complete investigation is made of the selfsimilar solutions of the boundary layer equation for the stream function with zero pressure gradient. They are a good description of the flow pattern in mixing layers since far from the separation point the latter is formed mainly under the effect of the boundary conditions and depends slightly on the initial conditions. The selfsimilar function $\Phi(\zeta; m)$ (ζ is the selfsimilar variable, and m the selfsimilarity parameter) satisfies a well-known third-order non-linear differential equation. It is successfully reduced to a first-order equation /1/, which enables us to investigate the behaviour of all the integral curves of $\Phi(\zeta; m)$ and, in particular, the examination of the question of the existence and uniqueness of the solutions of the two- and three-point problems that occur in the theory of displacement layers. For $m=1$ these are classical problems /2—4/ and the Blasius boundary layer problem and for $m=2$ the Goldstein problem for the wake /5/. The mixing layer encountered in the theory separations /6—11/ refers to the case $m \in (1, 2]$. The case $m=\infty$ occurs in the theory of non-stationary separation /12/.

From the viewpoint of the behaviour of the integral curves, the cases $m>1$ and $0<m\leqslant 1$ differ substantially. For $0<m\leqslant 1$ their pattern is reformed in such a manner that solutions describing the flows in mixing layers with reverse velocities do not occur. Examples of the latter are given in /13, 14/.

To a first approximation the flow in a mixing layer is described by the equation for the stream function

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\,\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=\lambda\frac{\partial^3\psi}{\partial y^3} \tag{1}$$

For an incompressible fluid $\lambda=1$. For a gas $\lambda=\theta/R^2(0)$ in the theory of local

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